Toward symplectic cohomology and Lagrangian Floer theory in toric degenerations

Bernd Siebert

Fachbereich Mathematik
Universität Hamburg

U. Penn.
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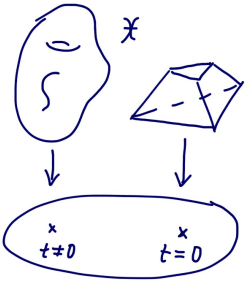
Motivation

Consider a toric degeneration (Gross/S. 2002–)

$$\pi : \mathfrak{X} \longrightarrow D$$

X_t: Something we want to study, e.g. a Calabi-Yau

X_0: a union of toric varieties



Discrete, "tropical" data $(B, \mathscr{P})$, essentially the union of the momentum polytopes of X_0, produce canonical, polarized such families via the smoothing algorithm (wall crossing, scattering etc.)

Pipe dream: Describe $\mathrm{Fuk}(X_t, \omega_t)$ by algebraic (log-) curve counts on X_0!

Today: Take some steps with the Tate curve

The tropical Morse category

(Abouzaid/Gross/S. 2007)

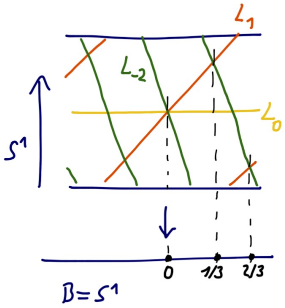
Gross 2008 (Clay book): Work out for the elliptic curve via toric degenerations.

Objects: $\mathbb{Z}$, thought of as powers of the ample line bundle in $D^b(\mathcal{O}_X)$, or as slope d of a Lagrangian L_d on the mirror side (morally, from $\check{X} \supset TB_0/\Lambda$).

Hom-spaces: $\mathrm{Hom}(L_i, L_j) = B\left(\frac{1}{|i-j|}\mathbb{Z}\right)$

Structure coefficients: Counting of tropical Morse trees

A tropical Morse tree is a piecewise linear version of an ordinary Morse tree.



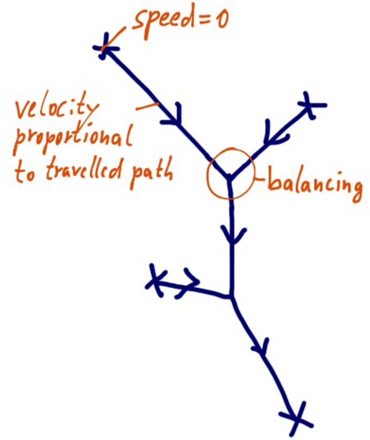
Tropical Morse trees

The leaves of the tree end at elements of the hom spaces. So these are the analogues of critical points in ordinary Morse theory

At a $1/d$-rational point the initial acceleration is d.

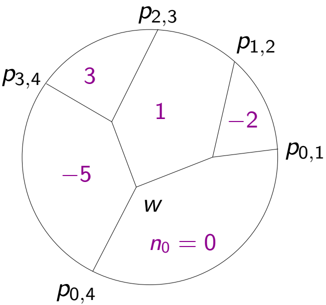
Leaves with label $i - j < 0$ mean the adjacent edge is contracted.

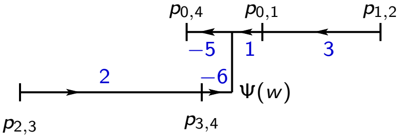
Under the presence of a wall structure the edges may bend at walls (in a specified fashion).

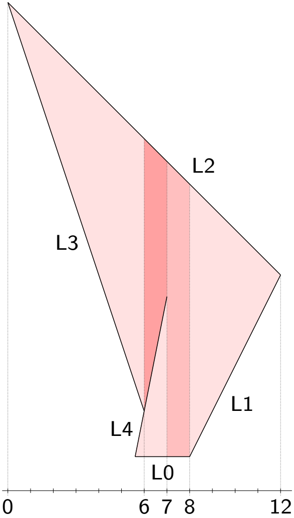


Tropical Morse trees and polygons

For the elliptic curve there is a direct interpretation of tropical Morse trees as (possibly degenerate) polygons bounded by the canonical Lagrangians L_i.





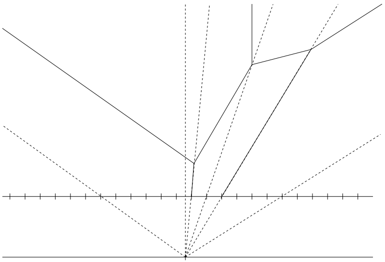


The cone trick

Taking the affine cone over B turns a tropical Morse tree into a truncated, but otherwise ordinary tropical curve, called a tropical coral. Unlike tropical Morse trees, the edges have piecewise rational slopes with piecewise constant integral velocity vectors.

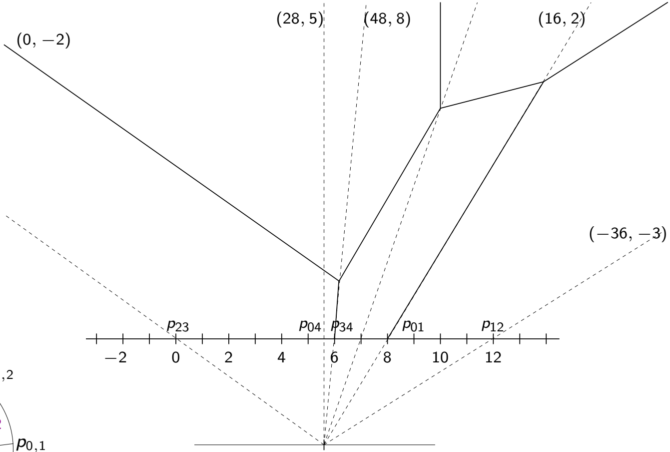
A positive end of a tropical Morse tree gives rise to an unbounded edge, going off to the top. The labelling rational point of B determines the asymptotic direction (but not the precise position).

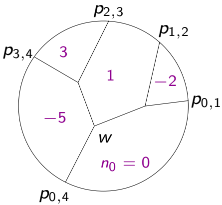
A negative end forces the tropical coral to have a vertex at the labelling point on the lower boundary of $\overline{C}B$. A single adjacent edge lies on a line through the origin.



With a non-trivial wall structure, the edges may interact with the walls lifted from B to $\overline{C}B$ (as in broken lines).

Example: A tropical coral





A symplectic challenge

Going over to $\overline{C}B$ on the complex side means going over to the total space of the ample line bundle. Indeed, broken lines on $\overline{C}B$ give rise to our theta functions! The labelling set is as expected.

On the symplectic side, $\overline{C}B$ describes the total space $\check{\mathfrak{X}}$ of the mirror degeneration. In this picture, tropical curves are tropicalizations of holomorphic curves in $\check{\mathfrak{X}}$ (or of a toric degeneration of $\check{\mathfrak{X}}$).

This suggest a description of the Fukaya category of X_t in terms of certain holomorphic curves in $\mathfrak{X}$ for any toric degeneration.

Suggestion (Abouzaid 2012): Go via $\mathrm{SH}^*(\mathfrak{X} \setminus X_0)$!

Indeed, one expects a close relationship between:

$$\mathrm{Fuk}(X_t) \longleftrightarrow \mathrm{WF}(\mathfrak{X} \setminus X_0) \longleftrightarrow \mathrm{SH}^*(\mathfrak{X} \setminus X_0).$$

The Tate curve

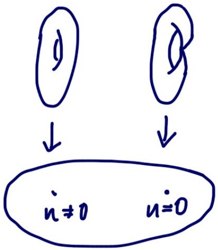
Now focus on the Tate curve. This example has no walls.
(Joint work with Hülya **Argüz**.)

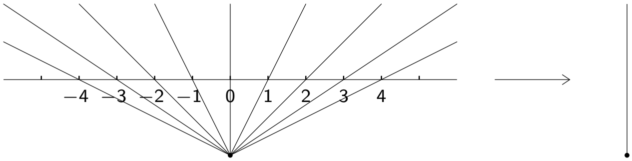
General fibre: $X_u = \mathbb{C}^*/\langle u \rangle$.

Central fibre: A cycle of b rational curves.

Construction: $B = S^1 = \mathbb{R}/(\mathbb{Z} \cdot b)$ of integral length b, decomposed into unit intervals.

- On the universal cover, do the toric construction: general fibre $\mathbb{C}^*$, central fibre $\bigcup_\infty \mathbb{P}^1$.
- restrict to $|t| < 1$;
- take the quotient by $\mathbb{Z}$ complex-analytically.





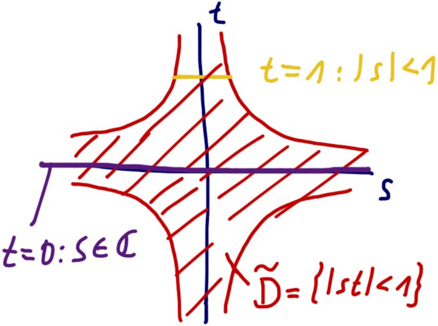
To the truncated cone

Induced degeneration $\tilde{\mathfrak{X}} \to \tilde{D}$ from the truncated cone generally agrees with the base change of $\mathfrak{X} \to D_u$ by

$$u \longmapsto st.$$

For constant $t \neq 0$ the degeneration (with parameter s) is the Tate curve with rescaled parametrization.

For $t \longrightarrow 0$ we zoom into the origin of the Tate curve $\mathfrak{X}$.



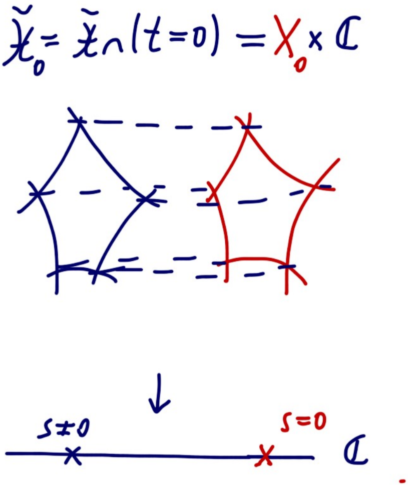
The central fibre for the truncated cone

The central fibre ($t = 0$) is now a product $\tilde{\mathfrak{X}}_0 = X_0 \times \mathbb{C}$.

The transition

$$\text{Tropical} \longrightarrow \text{Holomorphic}$$

works on $\tilde{X}_0$, and hence can be done purely algebraic-geometrically.



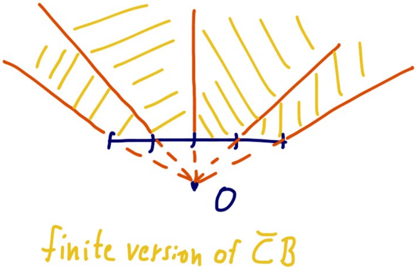
Reduction to a toric situation

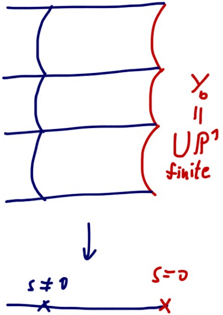
$\tilde{X}_0 = Y_0/\mathbb{Z}$, with $Y_0 \subset \mathcal{Y}$ and $\mathcal{Y}$ a toric variety.

Tropical corals are simply connected, as are rational curves.

We may thus lift to $\overline{C}\mathbb{R}$ and $\mathcal{Y}$, respectively.

We may also take only finitely many $\mathbb{P}^1$'s rather than an infinite chain.



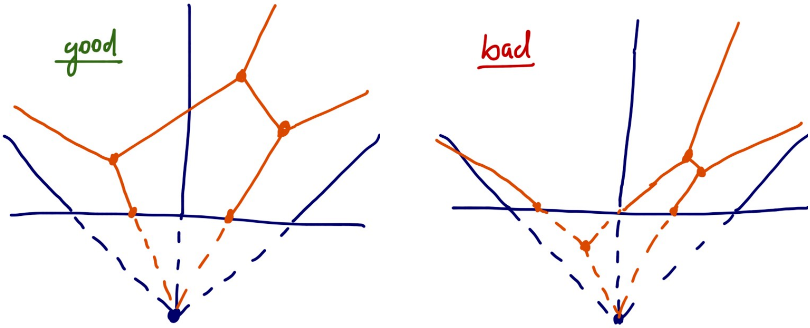


The lower boundary of $\overline{C}B$

The situation is almost as in Nishinou/S.: "Toric degenerations of toric varieties and tropical curves" (2004).

But $\overline{C}B$ has boundary and we have to address non-compactness.

Is the tropical count well-defined?



The stable range

For a tropical count we have to fix not only the asymptotics on the positive and negative ends, but also an asymptotic incidence condition for the positive ends (an affine line containing the respective unbounded edge).

Degree of a tropical coral: The tuple $\Delta = (\underline{\Delta}, \overline{\Delta})$ of asymptotic directions (determined by rational points on B), both positive and negative. For negative directions we always assume balancing at the lower boundary of $\overline{C}B$.

Stable range: Rescaling of $\overline{C}B$ (along with the incidence conditions) eventually makes all tropical corals of given degree and fulfilling the incidence conditions good.

Dimension count: For a fixed degree with n_+ positive and n_- negative ends, the moduli space of good tropical corals has dimension

$$n_+ - 1.$$

Counting tropical corals

Fix a degree $\Delta = (\underline{\Delta}, \overline{\Delta})$ and an incidence condition in the stable range.

$$N_{\Delta}^{\mathrm{trop}} = \frac{1}{\prod_i d_i} \sum_{\substack{\text{tropical coral } \Gamma \\ \text{fulfilling incidence}}} \mathrm{Mult}(\Gamma)$$

Mult(Γ): The usual multiplicity of a tropical curve (Mikhalkin).

d_i: the weights of the unbounded edges (denominators of the chosen rational point in B).

Log corals

Analyze stable log maps with tropicalization a log coral. Fix b from the rescaling of tropical corals.

A log coral is a map from a log-smooth curve over the standard log point $(\mathrm{pt}, \mathbb{C}^* \times \mathbb{N})$ to Y_0 (or to $\tilde{X}_0$):

$$\begin{array}{ccc} (C, \mathcal{M}_C) & \longrightarrow & (Y_0, \mathcal{M}_{Y_0}) \\ \downarrow & & \downarrow \\ (\mathrm{pt}, \mathbb{C}^* \times \mathbb{N}) & \xrightarrow{t \mapsto t^b} & (\mathrm{pt}, \mathbb{C}^* \times \mathbb{N}) \end{array}$$

$\mathcal{M}_{Y_0}$: The toric log structure.

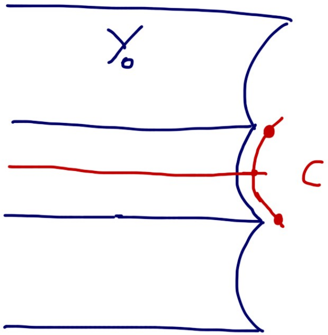
$(C, \mathcal{M}_C)$ has some marked points, all mapping to $X_0 = (s = 0) \subset Y_0$.

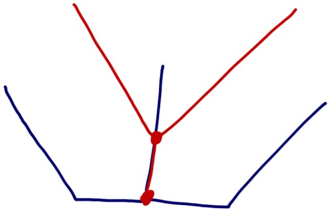
The non-compact components of C are copies of $\mathbb{C}$ projecting properly to the s-plane.

Example I

Fact: The tropicalization of a log coral is a tropical coral.

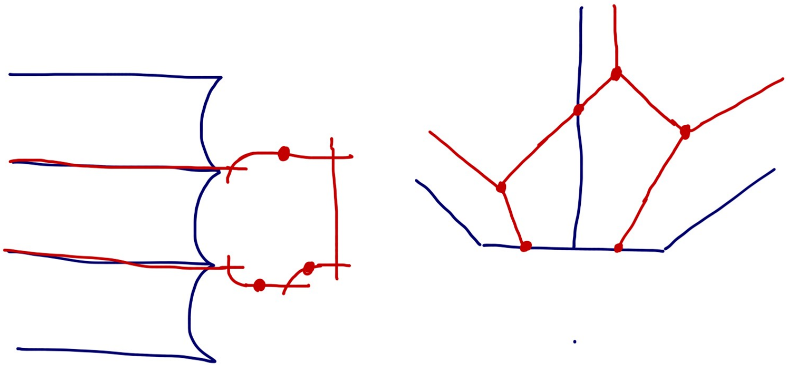
A tropically non-generic example:





Example II

A tropically generic example:



The red dots are the marked points. Tropically these give rise to the positive ends.

Asymptotically parallel

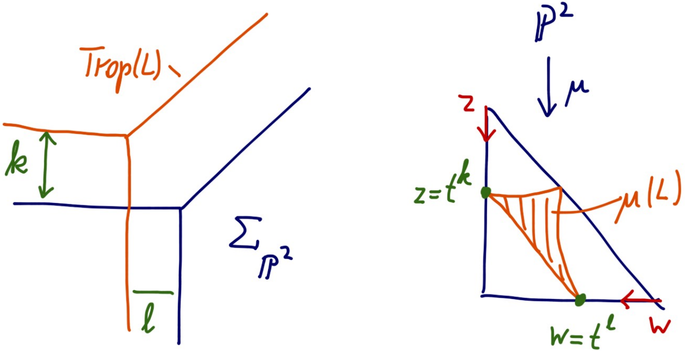
Important: This is an analogue of the stable range property. The definition essentially says that the unbounded components are constant in the X_0-direction.

The precise definition uses log geometry to make a stronger statement for those components of C mapping to the horizontal double locus of Y_0.

We can then impose incidence conditions both on the marked points and on the points over $s = \infty$.

Interpretation of the stable range property

Example: (Purely toric) Count lines on $\mathbb{P}^2$ through two points on toric divisors via the trivial degeneration $\mathbb{P}^2 \times \mathbb{C}$.



Conclusion: A counting problem on the original Tate curve $\mathcal{X} = \mathcal{Y}_1$ with marked points on the smooth locus of X_0 (wlog) only becomes well-defined when degenerating the incidences sufficiently fast to the double points of X_0.

The log counting problem

Theorem. (Argüz/S.) Choosing incidence conditions in the stable range, the count of log corals is well-defined (and each log coral in the count is unobstructed).

Moreover, it holds $\boxed{N_\Delta^{\text{log}} = N_\Delta^{\text{trop}}}$.

Observation: The unbounded components in log corals do not carry any information other than the intersection point with the adjacent component. Removing these components lead to counts of certain punctured Gromov-Witten invariants of $X_0 = (s = 0) \subset Y_0$.

A theory of punctured GW invariants is currently under development (Abramovich/Chen/Gross/S.).

An algebraic-geometric Fukaya category

We now have an interpretation of tropical Morse trees in terms of certain log Gromov-Witten invariants of $\mathfrak{X}$.

Proposal I:

Use log corals directly to define the Fukaya category of X_t.

More symplectically:

Proposal II:

Show that our log corals compute (higher structures on) symplectic cohomology

Symplectic cohomology

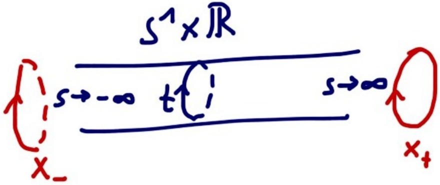
For non-compact symplectic manifolds M with cylindrical ends $N \times \mathbb{R}$, N a contact manifold. The construction (but hopefully not the result) depends on the choice of a Hamiltonian H with $H' \to \infty$ near the ends. Invariance is typically a challenge.

Generators: Time-1 periodic orbits of H. Near the ends these are in bijection with closed orbits of any period of the Reeb vector field of N.

Differentials: Cylindrical solutions of the Floer equation

$$\partial_s u + J(\partial_t u - X_H) = 0$$

with $\lim_{s \to \pm\infty} u$ a time-1 periodic orbit of H.

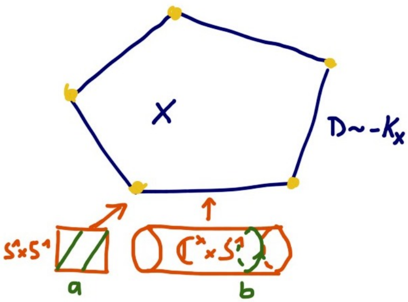


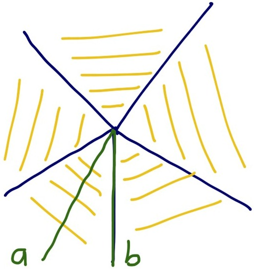
Some relevant known cases

I. Hamiltonian mapping tori (Fabert 2006):
$M_\phi = M \times [0,1] \times \mathbb{R}/(p,1,r) \sim (\phi(p),0,r)$ for $\phi : M \to M$ a Hamiltonian symplectomorphism. Obtain Floer homology for all powers ϕ^T.

The Tate curve is topologically the mapping torus of a Dehn twist.

II. Log Calabi-Yau surfaces (Pascaleff 2013):
Identifies a basis of SH^0 with integral points in the dual intersection complex.





First challenges

I. What is the good symplectic structure on $\mathcal{X} \setminus X_0$ to work with?

Candidates: 1) From embedding into $\mathbb{P}^2$.

2) A Ricci-flat one.

3) A symplectization of an asymptotic contact manifold, $r \to 0$.

4) A symplectic mapping torus for a linear realization of the Dehn twist.

Maybe not so relevant?

II. What is the natural Hamiltonian to use?

Guess:

$$H'(r) \xrightarrow{r\to 0} \infty, \quad H'(r) \xrightarrow{r\to 1} 0, \quad H'' < 0.$$

This leads to wrapping only close to X_0 (zoom into the origin).

The real challenges:

Once properly set up, show that our **log corals** deform uniquely to **solutions of the Floer equation**:

- First use a (log-) holomorphic deformation argument to deform the log coral to $0 < |t| \ll 1$.
  Critical: Replace the non-compact components by discs bounding small circles around X_0 ($\leftrightarrow$ Pascaleff's generators)
- Use a wrapping trick to transform holomorphic curves to something close (or equal?) to a solution of the Floer equation (cf. work of Luis Diogo). Then apply an implicit function theorem to get to a proper solution.
- A compactness argument should take care of the converse.

Question: Where does the stable range property enter? Guess: Compactness.

Conclusions

We have sketched, at the example of the Tate degeneration of elliptic curves, how log Gromov-Witten techniques may be used for an algebraic-geometric definition of the Fukaya category.

To make contact with symplectic geometry, subtle problems remain to relate log coral counts to symplectic cohomology, but these may be solvable.

An alternative route could be taken by the wrapped Fukaya category. A natural generator is the positive real locus, lying over $u \in \mathbb{R}_{>0} \subset D$.

The picture also suggests that the symplectic monodromy transformation coming from a maximal degeneration should play a fundamental role on the symplectic side of mirror symmetry. It generates the Fukaya category from an SYZ section!